Worcester County Mathematics League

Varsity Meet 4
February 25, 2015

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 – February 25, 2014
Round 1: Number Theory

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Find the greatest common factor (GCF) of 1665 and 2430.

2. Find all ordered pairs of natural numbers (a, b), where $a < b$, for which the greatest common factor of a and b is 12, and the product of a and b is 25200.

3. In base–16, the digits are (in increasing order):

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.$$

That is, A represents the base–10 quantity 10, B represents the base–10 quantity 11, etc. Write the base–16 number $3D5_{16}$ as a number in base–8.

ANSWERS

(1 pt.) 1. ________________________

(2 pts.) 2. ________________________

(3 pts.) 3. __________________ (base 8)

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 – February 25, 2014
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Kyle buys a new calculator and two math team shirts for \$31.87. If the calculator cost \$13.99, how much would it cost for someone to buy 3 math team shirts?

2. The average (arithmetic mean) of x and y is 20, and the average of y and 20 is $\frac{z}{2}$. What is the average of x and z?

3. Solve $\sqrt{13x-3} = 2 + \sqrt{9x-11}$

ANSWERS

(1 pt.) 1. \$ ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 – February 25, 2014
Round 3: Geometry

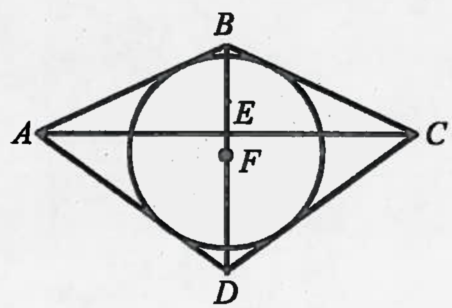
All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. $\overrightarrow{MQ}$ bisects $\angle LMN$,
   $\overrightarrow{MR}$ bisects $\angle LMQ$, and
   $\overrightarrow{MS}$ bisects $\angle LMR$.

If the measure of $\angle SMR$ is $(4x - 6)°$ and the measure of $\angle SML$ is $(2x + 1)°$, find the measure of $\angle LMN$.

2. An isosceles trapezoid has two interior angles with measure $135°$, and the shorter of its two bases has length 23. If the area of the trapezoid is 210 square units, find the length of its other base.

3. In the figure, $ABCD$ is a kite, where $AB = BC = 13$, $CD = DA = 15$, and $AC = 24$. The diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. F is the center is the kite's inscribed circle. Find the length of $\overline{EF}$.



ANSWERS

(1 pt.) 1. ______________________ °

(2 pts.) 2. ____________________ units

(3 pts.) 3. ____________________ units

Varsity Meet 4 – February 25, 2014
Round 4: Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Solve for x:

$$(\sqrt{8})^{\frac{4}{3}} = 2^x$$

2. Simplify the expression:

$$\frac{\sqrt{5} - \frac{1}{\sqrt{5}}}{\frac{1 - \sqrt{5}}{\sqrt{5}}}$$

3. Simplify:

$$\frac{\log_9 10000}{\log_9 1000} \cdot \frac{\log_8 100}{\log_8 10}$$

ANSWERS

(1 pt.) 1. ____________________

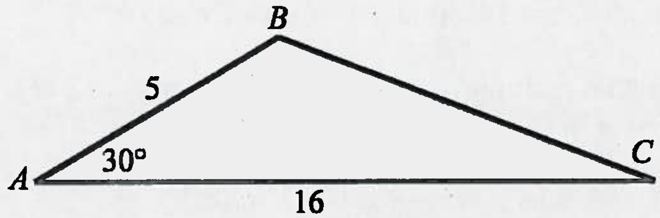
(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 4 – February 25, 2014
Round 5: Trigonometry

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. In ΔABC, $AC = 5$, $AB = 16$, and the measure of $\angle A$ is $30°$. Find the area of ΔABC.



2. Find all values of x (in degrees), where $0 \leq x < 360$, that satisfy:

$$\sin(x°) = -\sqrt{3}\cos(x°)$$

3. If $\sin x - \cos x = \frac{3}{5}$, what is the value of $\sin(2x)$?

ANSWERS

(1 pt.) 1. ________________ square units

(2 pts.) 2. _________________________ °

(3 pts.) 3. _________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 4 – February 25, 2014

Team Round

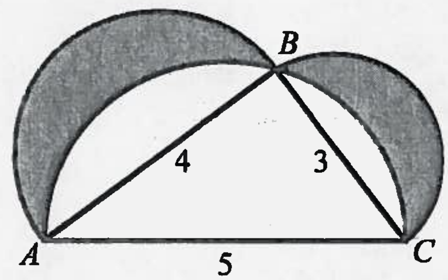
All answers must be in simplest exact form, and be written on the separate team answer sheet

1. If $2^x 4^y = 512$ and $5^{x-y} = 125$, find the values of x and y.

2. Find the least common multiple of the two base–5 numbers 1200_5 and 144_5. Give your answer as a number in base–10.

3. Find all values of x for which $(x^2 - 6x + 4)^2 = 16$.

4. In the figure, arc BC is a semicircle with diameter $\overline{BC}$, arc AB is a semicircle with diameter $\overline{AB}$, and arc AC is a semicircle with diameter AC. Find the sum of the areas of the two shaded regions, and give your answer in <u>exact</u> simplified form.



5. If $b = \log_3 x$, find all values of x that satisfy:

$$\log_b(-6 + \log_3 x^5) = 2$$

6. A, B, C, D, and E are five distinct points that lie on a circle. How many different convex polygons can be formed whose vertices are selected from these five points?

7. Find, in degrees, all values of x, $0 \le x < 360$, for which

$$4^{\sin^2 x} \cdot 4^{\cos^2 x} \cdot 4^{\tan^2 x} = 4^2$$

8. Let $A = \{1, 4, 9, 16, \ldots\}$ be the set of all perfect squares;
   $B = \{1, 2, 3, 4, 5, 7, \ldots\}$ be the set of all positive integers with at most three positive factors; and
   $C = \{400, 401, \ldots, 899, 900\}$ be the set of integers from 400 to 900.

Find $(A \cap B) \cap C$.

9. Let $f(x) = x + 1$ and $g(x) = 2x - 3$. The operation $\otimes$ is defined for all real numbers a and b by

$$a \otimes b = 2a + b - ab$$

What is the value of $f(3) \otimes g(5)$?

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Answers

Round 1: Number Theory

1. 45
2. $(12, 2100)$ and $(84, 300)$
3. 1725

Bartlett, Hudson, Auburn

Round 2: Algebra 1

1. 26.82
2. 30
3. $\{3, 4\}$ or $3, 4$ or $x = 3, 4$

Bromfield, Auburn, Westborough

Round 3: Geometry

1. 64
2. 37
3. 1.5 or $1\frac{1}{2}$ or $\frac{3}{2}$

UPCS, Tahanto, QSC

Round 4: Logs, Exponents, and Radicals

1. 2
2. $-1 - \sqrt{5}$
3. $\frac{8}{3}$ or $2\frac{2}{3}$ or $2.\overline{6}$

QSC, Doherty, Assabet Valley

Round 5: Trigonometry

1. 20
2. $\{120, 300\}$ or $120, 300$ or $x = 120, 300$
3. $\frac{16}{25}$ or 0.64

Oxford, Tahanto, St. John's

WORCESTER COUNTY MATHEMATICS LEAGUE

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ANSWERS: Team Round

1. $x = 5$, $y = 2$

2. 1225

3. $\{0, 2, 4, 6\}$ or $0, 2, 4, 6$ or $x = 0, 2, 4, 6$

4. 6

5. $\{9, 27\}$ or $9, 27$ or $x = 9, 27$

6. 16

7. $\{45, 135, 225, 315\}$ or $45, 135, 225, 315$ or $x = 45, 135, 225, 315$

8. $\{529, 841\}$ [$\{23^2, 29^2\}$ is <u>not</u> acceptable]

9. -13

WORCESTER COUNTY MATHEMATICS LEAGUE

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Team Round Answer Sheet

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM, AND BE WRITTEN ON THIS TEAM ANSWER SHEET.

(2 points each)

1. $x =$ ________ $y =$ ________

2. __________________ **(base 10)**

3. ________________________

4. ________________________

5. ________________________

6. ________________________

7. ________________________°

8. ________________________

9. ________________________

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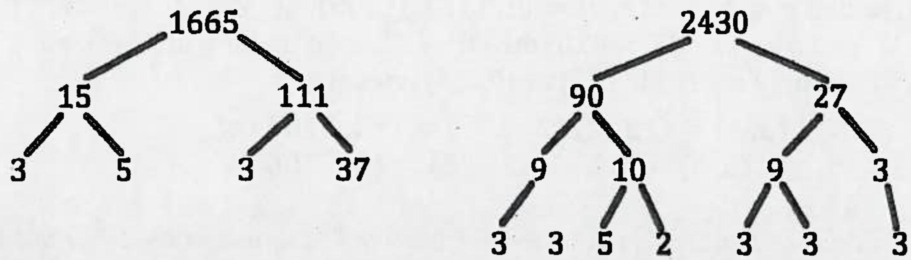
Solutions

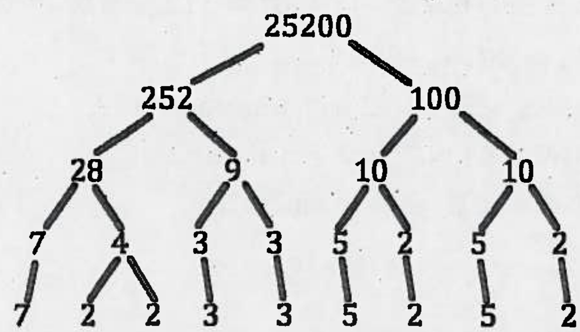
Round 1: Number Theory

1. First, find the prime factorization of each number. Using a factor tree, for example:



That is, the two numbers' prime factorizations are $1665 = 3^2 \cdot 5 \cdot 37$ and $2430 = 2 \cdot 3^5 \cdot 5$. Their common prime factors are 3^2 and 5, so their GCF is $3^2 \cdot 5 = \mathbf{45}$.

2. Solution 1: First, find the prime factorization of 25200:



That is, $25200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$. Since the GCF of a and b is 12, each must contain $12 = 2^2 \cdot 3$ as a factor. Then we can write $a = 2^2 \cdot 3 \cdot m$ and $b = 2^2 \cdot 3 \cdot n$ for some integers m and n. Because $ab = 25200$, this implies

$$(2^2 \cdot 3 \cdot m) \cdot (2^2 \cdot 3 \cdot n) = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$$
$$2^4 \cdot 3^2 \cdot m \cdot n = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$$
$$m \cdot n = 5^2 \cdot 7$$

This means that m and n must collectively comprise the prime factors 5, 5, and 7. If both m and n had one of the 5s as a factor, then a and b would both have a common factor of 5, and their GCF would not be 12, so this possibility is excluded. The remaining options are that $m = 1$ and $n = 5^2 \cdot 7 = 175$, or $m = 7$ and $n = 25$ (remembering that $a < b$ was specified).

In the first case, $a = 12 \cdot 1 = 12$ and $b = 12 \cdot 175 = 2100$. In the second case, $a = 12 \cdot 7 = 84$ and $b = 12 \cdot 25 = 300$. So the two solutions are

$$(\mathbf{12, 2100}) \text{ and } (\mathbf{84, 300})$$

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Solution 2: Since $GCF(a,b) = 12$, $a = 12x$ and $b = 12y$ for some integers x and y. Then $25200 = ab = 12x \cdot 12y = 144xy$, which implies $xy = 25200 \div 144 = 175$.

As the prime factorization of 175 is $175 = 5^2 \cdot 7$, the possible integer values of x and y (where $x < y$, so that $a < b$) are $(x,y) = (1,175), (5,35)$, or $(7,25)$. However, if $a = 12 \cdot 5 = 60$ and $b = 12 \cdot 35 = 420$, then the GCF of a and b will be 60, not 12. Therefore (x,y) can only equal $(1,175)$ or $(7,25)$, meaning

$$(a,b) = (12 \cdot 1, 12 \cdot 175) = (\mathbf{12,2100}) \text{ or}$$
$$(a,b) = (12 \cdot 7, 12 \cdot 25) = (\mathbf{84,300})$$

3. The base–16 digit D represents the base–10 quantity 13, so the (base–10) value of $3D5_{16}$ is

$$\begin{aligned} &3 \cdot 16^2 + 13 \cdot 16 + 5 \\ &= 3 \cdot 256 + 13 \cdot 16 + 5 \\ &= 768 + 208 + 5 \\ &= 981 \end{aligned}$$

To write this in base 8, first note that $8^2 = 64$, $8^3 = 512$, and $8^4 = 4096$. Then:

There is no "4096"s digit, as $981 < 4096$.

The "512"s digit is 1, as $981 \div 512 = 1$, with remainder 469.

The "64"s digit is 7, as $469 \div 64 = 7$, with remainder 21.

The "8"s digit is 2, as $21 \div 8 = 2$, with remainder 5.

The "1"s digit is 5.

Then the number is $\mathbf{1725_8}$

Round 2: Algebra 1

1. By the given information, the cost of two shirts is $\$31.87 - \$13.99 = \$17.88$. This means the cost of one shirt is $\frac{1}{2} \cdot \$17.88 = \$8.94$, so the cost of three shirts is $3 \cdot \$8.94 =$ $\mathbf{\$26.82}$.

2. The average of x and y being 20 gives the equation $\frac{x+y}{2} = 20$, i.e. $x + y = 40$. The average of y and 20 being $\frac{z}{2}$ gives $\frac{y+20}{2} = \frac{z}{2}$, meaning $y + 20 = z$.
To find the average of x and z, first isolate x, to find that $x = 40 - y$. Then

$$\frac{x+z}{2} = \frac{(40-y)+(y+20)}{2} = \frac{60}{2} = \mathbf{30}$$

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3. Begin by squaring both sides of the given equation $\sqrt{13x-3} = 2 + \sqrt{9x-11}$ to get

$$13x - 3 = 4 + 2 \cdot 2\sqrt{9x-11} + 9x - 11$$
$$13x - 3 = 9x - 7 + 4\sqrt{9x-11}$$
$$4x + 4 = 4\sqrt{9x-11}$$
$$x + 1 = \sqrt{9x-11}$$

Then squaring both sides again gives:

$$x^2 + 2x + 1 = 9x - 11$$
$$x^2 - 7x + 12 = 0$$
$$(x-3)(x-4) = 0$$

Then the two possible solutions are $x = 3$ and $x = 4$. Since we squared both sides of the equation as one of our steps, we must check for extraneous roots. However in this case, both 3 and 4 satisfy the original equation:

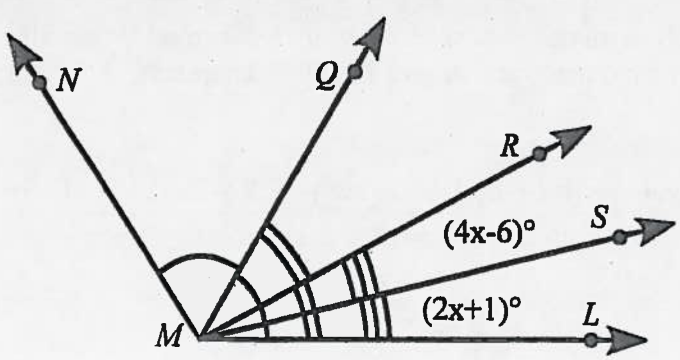
$$\sqrt{13 \cdot 3 - 3} = 2 + \sqrt{9 \cdot 3 - 11}$$
$$\sqrt{39 - 3} = 2 + \sqrt{27 - 11}$$
$$\sqrt{36} = 2 + \sqrt{16}$$
$$6 = 2 + 4$$

$$\sqrt{13 \cdot 4 - 3} = 2 + \sqrt{9 \cdot 4 - 11}$$
$$\sqrt{52 - 3} = 2 + \sqrt{36 - 11}$$
$$\sqrt{49} = 2 + \sqrt{25}$$
$$7 = 2 + 5$$

Therefore the two solutions are $x = 3, 4$.

Round 3:

1. Solution 1: Since $\overrightarrow{MS}$ bisects $\angle RML$, $4x - 6 = 2x + 1$.



Solving this, we have $2x = 7$, meaning $x = 3.5$. Then the measure of $\angle SML$ (and therefore $\angle RMS$, as well) is $2 \cdot 3.5 + 1 = 7 + 1 = 8$ degrees. This means the measure of $\angle RML$ is $2 \cdot 8° = 16°$, the measure of $\angle QML$ is $2 \cdot 16° = 32°$, and the measure of $\angle NML$ is $2 \cdot 32° = 64°$.

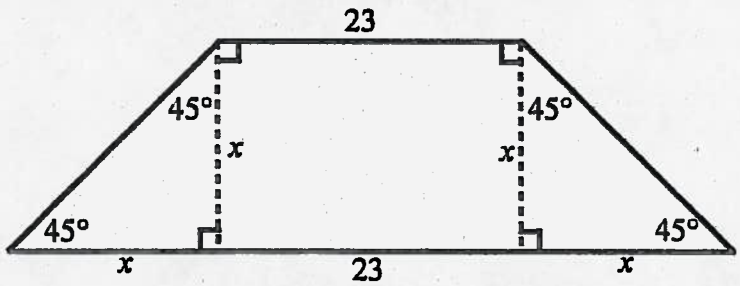
Solution 2: $m\angle SML = m\angle RMS = 4x - 6$, meaning $m\angle RML = 2(4x-6) = 8x - 12$. $m\angle QMR = m\angle RML = 8x - 12$, meaning $m\angle QML = 2(8x-12) = 16x - 24$. And $m\angle NMQ = m\angle QML = 16x - 24$, so that $m\angle LMN = 2(16x-24) = 32x - 48$.

On the other hand, $m\angle SML = m\angle RMS = 2x + 1$, meaning
$m\angle RML = 2(2x+1) = 4x + 2$.
$m\angle QMR = m\angle RML = 4x + 2$, meaning $m\angle QML = 2(4x+2) = 8x + 4$. And $m\angle NMQ = m\angle QML = 8x + 4$, so that $m\angle LMN = 2(8x+4) = 16x + 8$.

$$m\angle LMN = 32x - 48 = 16x + 8$$
$$16x = 56$$
$$x = \frac{56}{16} = 3.5$$

This gives $m\angle LMN = 32(3.5) - 48 = 112 - 48 = 64$ degrees.

2. Let x be the trapezoid's height. Then, if we draw altitudes at two of the trapezoid's vertices (shown as dashed lines), they form 45° angles, as $135° - 90° = 45°$.



Consequently, the third angle of each of the triangles formed by the altitudes must also be 45°, since the sum of the triangles' angles is 180°. Therefore, the triangles are isosceles, as shown.

The trapezoid's area is $\frac{1}{2}h(b_1 + b_2) = \frac{1}{2}x\big(23 + (23 + 2x)\big) = \frac{1}{2}x(46 + 2x)$
$= x(23 + x) = x^2 + 23x = 210$. Then:

$$x^2 + 23x - 210 = 0$$
$$(x + 30)(x - 7) = 0$$
$$x = -30, 7$$

Since x must be positive, $x = 7$, and $23 + 2x = 23 + 14 = 37$

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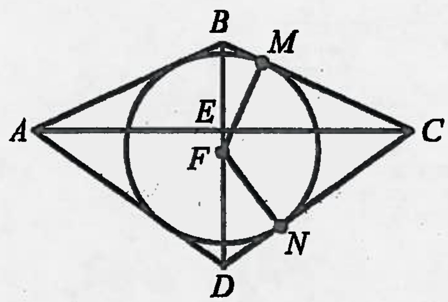
3. Since $ABCD$ is a kite, $\overline{BD}$ bisects $\overline{AC}$, meaning $AE = EC = 12$. Then by the Pythagorean Theorem, $BE = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$, and $DE = \sqrt{15^2 - 12^2} = \sqrt{225 - 144} = \sqrt{81} = 9$. Alternatively, we could recognize ΔBEC as a 5-12-13 right triangle to find that $BE = 5$, and ΔDEC scaled 3-4-5 right triangle, specifically 9-12-15, to find that $DE = 9$.



Since the diagonals of a kite intersect at right angles, and the radius to a point on the circle is perpendicular to the circle's tangent line, we see that $\angle BMF \cong \angle BEC$. Then ΔBMF and ΔBEC have two pairs of congruent angles, from which it follows that $\angle BCE \cong \angle BFM$ as well, meaning ΔBMF and ΔBEC are similar. Therefore $\frac{BF}{MF} = \frac{BC}{EC}$, i.e. $\frac{5+x}{r} = \frac{13}{12}$ (where $x = EF$ and $r = MF$), which gives $60 + 12x = 13r$.

Furthermore, ΔDNF is similar to ΔDEC, so $\frac{DF}{NF} = \frac{DC}{EC}$, i.e. $\frac{9-x}{r} = \frac{15}{12}$, which gives $108 - 12x = 15r$. We solve this system by adding the two equations, to get

$$168 = 28r$$
$$r = 6$$

Substituting this value in the previous equation,

$$108 - 12x = 15 \cdot 6 = 90$$
$$12x = 108 - 90 = 18$$
$$x = \frac{18}{12} = \mathbf{1.5}$$

Round 4:

1. Solution 1: Since $8 = 2^3$ and $\sqrt{x} = x^{\frac{1}{2}}$, we have:

$$(\sqrt{8})^{\frac{4}{3}} = \left(\sqrt{2^3}\right)^{\frac{4}{3}} = \left((2^3)^{\frac{1}{2}}\right)^{\frac{4}{3}} = 2^{3\cdot\frac{1}{2}\cdot\frac{4}{3}} = 2^{\frac{12}{6}} = 2^2$$

Therefore $x = \mathbf{2}$.

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Solution 2:

$$(\sqrt{8})^{\frac{4}{3}} = (2\sqrt{2})^{\frac{4}{3}} = \sqrt[3]{(2\sqrt{2})^4} = \sqrt[3]{2^4 \cdot (\sqrt{2})^4} = \sqrt[3]{16 \cdot 4} = \sqrt[3]{64} = 4$$

Since $4 = 2^2$, $x = \mathbf{2}$.

2.

$$\frac{\sqrt{5} - \frac{1}{\sqrt{5}}}{\frac{1-\sqrt{5}}{\sqrt{5}}} = \frac{\sqrt{5} \cdot \frac{\sqrt{5}}{\sqrt{5}} - \frac{1}{\sqrt{5}}}{\frac{1-\sqrt{5}}{\sqrt{5}}} = \frac{\frac{5}{\sqrt{5}} - \frac{1}{\sqrt{5}}}{\frac{1-\sqrt{5}}{\sqrt{5}}} = \frac{\frac{4}{\sqrt{5}}}{\frac{1-\sqrt{5}}{\sqrt{5}}}$$

$$= \frac{4}{\sqrt{5}} \cdot \frac{\sqrt{5}}{1-\sqrt{5}} = \frac{4}{1-\sqrt{5}} = \frac{4}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = \frac{4(1+\sqrt{5})}{1-5}$$

$$= \frac{4(1+\sqrt{5})}{-4} = -(1+\sqrt{5}) = \mathbf{-1-\sqrt{5}}$$

3. Using the change of base formula, we have:

$$\frac{\log_9 10000}{\log_9 1000} \cdot \frac{\log_8 100}{\log_8 10} = \log_{1000} 10000 \cdot \log_{10} 100$$

To evaluate $\log_{1000} 10000$, let $x = \log_{1000} 10000$, meaning $1000^x = 10000$. That is,

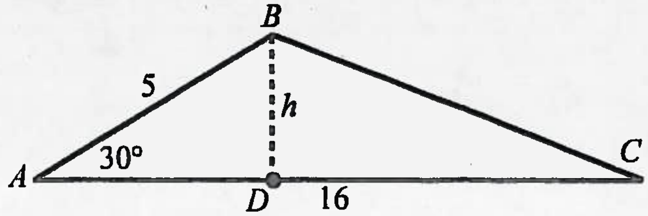
$$(10^3)^x = 10^4$$
$$10^{3x} = 10^4$$
$$3x = 4$$
$$x = \frac{4}{3}$$

And $\log_{10} 100 = 2$, as $10^2 = 100$. So altogether,

$$\log_{1000} 10000 \cdot \log_{10} 100 = \frac{4}{3} \cdot 2 = \mathbf{\frac{8}{3}}$$

Round 5:

1. Solution 1: To compute the area, we first find the height h:



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Since $\angle ADB$ is a right angle, we have

$$\frac{h}{5} = \sin(30°) = \frac{1}{2}$$

$$h = \frac{5}{2} = 2.5$$

Then the area of ΔABC is $\frac{1}{2}bh = \frac{1}{2} \cdot 16 \cdot 2.5 = \mathbf{20}$ square units.

2. $\sin(x) = -\sqrt{3}\cos(x)$ implies that

$$-\sqrt{3} = \frac{\sin x}{\cos x} = \tan x$$

To find all such values of x, first find the reference angle y, where $0 \le y \le 90$, and $\tan(y°) = +\sqrt{3}$

From special right triangles—specifically a 30-60-90 triangle, we have $y = 60$. To find x, note that $\tan x$ is negative only in the second and fourth quadrants, meaning the two solutions to $\tan x = -\sqrt{3}$ are:

$$x = 180 - y = 180 - 60 = \mathbf{120} \text{ and } x = 360 - y = 360 - 60 = \mathbf{300}.$$

3.

$$\sin x - \cos x = \frac{3}{5}$$

$$(\sin x - \cos x)^2 = \left(\frac{3}{5}\right)^2$$

$$\sin^2 x - 2\sin x\cos x + \cos^2 x = \frac{9}{25}$$

$$(\sin^2 x + \cos^2 x) - 2\sin x\cos x = \frac{9}{25}$$

$$1 - \sin(2x) = \frac{9}{25}$$

(This step follows from the identities $\sin^2 x + \cos^2 x = 1$, and $2\sin x\cos x = \sin(2x)$.)

$$\sin(2x) = 1 - \frac{9}{25} = \mathbf{\frac{16}{25}}$$

Team Round

1. Solution 1: Since $4 = 2^2$, we have $2^x 4^y = 2^x(2^2)^y = 2^{x+2y} = 512$. Since $512 = 2^9$, this implies $x + 2y = 9$.

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Similarly, $125 = 5^3$, so $5^{x-y} = 5^3$ gives $x - y = 3$. Subtracting this equation from $x + 2y = 9$, we get $3y = 6$, meaning $y = 2$. Then $x - 2 = 3$, giving $x = 5$.

Solution 2:

$$2^x 4^y = 512$$
$$2^x (2^2)^y = 2^9$$
$$2^{x+2y} = 2^9$$
$$x + 2y = 9$$

$$5^{x-y} = 125$$
$$5^{x-y} = 5^3$$
$$x - y = 3$$

$$x + 2y = 9$$
$$\underline{-\ [x - y = 3]}$$
$$3y = 6$$
$$y = 2$$

$$x - (2) = 3$$
$$x = 5$$

2. The value of 1200_5 in base 10 is $1 \cdot 5^3 + 2 \cdot 5^2 = 125 + 50 = 175$. The value of 144_5 in base 10 is $1 \cdot 5^2 + 4 \cdot 5 + 4 = 25 + 20 + 4 = 49$. To find the LCM of these two numbers, we find their prime factorizations: $175 = 5^2 \cdot 7$ and $49 = 7^2$. The LCM contains all prime factors in either number, so it equals $5^2 \cdot 7^2 = 1225$.

3. Solution 1: Subtract 16 from each side of the equation, then factor:

$$(x^2 - 6x + 4)^2 = 16$$
$$(x^2 - 6x + 4)^2 - 16 = 0$$
$$(x^2 - 6x + 4)^2 - 4^2 = 0$$
$$(x^2 - 6x + 4 + 4)(x^2 - 6x + 4 - 4) = 0$$
$$(x^2 - 6x + 8)(x^2 - 6x) = 0$$
$$(x - 4)(x - 2)(x - 6)x = 0$$

Then $x = 0, 2, 4, 6$

Solution 2: Rather then factor the entire degree 4 polynomial, we can take the square root (including a plus/minus sign) of each side to get two quadratics:

$$\sqrt{(x^2 - 6x + 4)^2} = \pm\sqrt{16}$$

$x^2 - 6x + 4 = 4$	or	$x^2 - 6x + 4 = -4$
$x^2 - 6x = 0$	or	$x^2 - 6x + 8 = 0$
$x(x - 6) = 0$	or	$(x - 4)(x - 2) = 0$

Therefore, $x = \mathbf{0, 2, 4, 6}$

4. Let w equal the area of the triangle with vertices A, B, and C, x equal the area of the semicircle with diameter 3, y equal the area of the semicircle with diameter 4, and z equal the area of the semicircle with diameter 5. Then the area of the shaded regions equals $w + x + y - z$.

To find w, note that $3^2 + 4^2 = 5^2$, meaning ΔABC is a right triangle. Hence, its legs are its base and height respectively, and $w = \frac{1}{2} \cdot 3 \cdot 4 = 6$

Since the area of a semicircle is given by $\frac{1}{2}\pi r^2$ (and $r = \frac{1}{2}d$), we have:

$$x = \frac{1}{2}\pi\left(\frac{3}{2}\right)^2 = \frac{\pi}{2} \cdot \frac{9}{4} = \frac{9}{8}\pi$$

$$y = \frac{1}{2}\pi\left(\frac{4}{2}\right)^2 = \frac{\pi}{2} \cdot \frac{16}{4} = \frac{16}{8}\pi$$

$$z = \frac{1}{2}\pi\left(\frac{5}{2}\right)^2 = \frac{\pi}{2} \cdot \frac{25}{4} = \frac{25}{8}\pi$$

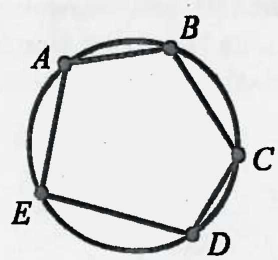
Then $w + x + y - z = 6 + \frac{9}{8}\pi + \frac{16}{8}\pi - \frac{25}{8}\pi = \mathbf{6}$

5. $\log_b(-6 + \log_3 x^5) = 2$ implies that $-6 + \log_3 x^5 = b^2 = (\log_3 x)^2$. Since $\log_3 x^5 = 5\log_3 x$, we have

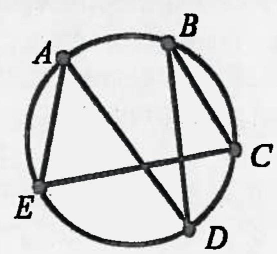
$$-6 + 5\log_3 x = (\log_3 x)^2$$
$$(\log_3 x)^2 - 5\log_3 x + 6 = 0$$
$$(\log_3 x - 3)(\log_3 x - 2) = 0$$

Then $\log_3 x = 3$ or 2. These two possibilities respectively give $x = 3^3 = \mathbf{27}$ and $x = 3^2 = \mathbf{9}$.

6. First, note that for a number of (3 or more) points on a circle, there is exactly one convex polygon that can be formed with those points as its vertices. (Indeed, any other polygon formed from these vertices must self-intersect, as shown in the figure.)



Polygon *ABCDE* is convex (left), but polygon *ADBCE* is not (right).



Then the number of convex polygons is equal to the number of ways of selecting three or more vertices from the set $\{A, B, C, D, E\}$. The number of ways of selecting three vertices is:

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$${}_5C_3 = \frac{5!}{3!\,(5-3)!} = \frac{5\cdot 4}{2\cdot 1} = \frac{20}{2} = 10$$

The number of ways of selecting four vertices is:

$${}_5C_4 = \frac{5!}{4!\,(5-4)!} = \frac{5}{1} = 5$$

And the number of ways of selecting five vertices is:

$${}_5C_5 = \frac{5!}{5!\,(5-5)!} = 1$$

Then altogether there are $10 + 5 + 1 = \mathbf{16}$ possible polygons.

7.

$$4^{\sin^2 x} \,.\, 4^{\cos^2 x} \,.\, 4^{\tan^2 x} = 4^2$$

$$4^{\sin^2 x + \cos^2 x + \tan^2 x} = 4^2$$

$$(\sin^2 x + \cos^2 x) + \tan^2 x = 2$$

$$1 + \tan^2 x = 2$$

$$\tan^2 x = 1$$

$$\tan x = \pm\sqrt{1} = \pm 1$$

To find all such values of x, we first find the reference angle y, $0 \le y \le 90$, such that $\tan y = 1$. From knowledge of a 45-45-90 special right triangle, we have $y = 45$.

Since $\tan x$ can equal positive or negative 1, there will be one solution in each quadrant, namely:

$$x = y = \mathbf{45}$$

$$x = 180 - y = 180 - 45 = \mathbf{135}$$

$$x = 180 + y = 180 + 45 = \mathbf{225}$$

$$x = 360 - y = 360 - 45 = \mathbf{315}$$

8. First note that B consists only of the number 1, all prime numbers, and the squares of prime numbers. (If a number n had two distinct prime factors p and q, then it would have at least four distinct factors: 1, p, q, and pq.) Then $A \cap B$ consists of 1, and all squares of prime numbers:

$$A \cap B = \{1, 4, 9, 25, 49, 121, \ldots\}$$

To find all such "prime squares" between 400 and 900, note that $400 = 20^2$ and $900 = 30^2$. Then $(A \cap B) \cap C$ must consist of the squares of all primes between 20 and 30. The only such primes are 23 and 29, meaning:

$$(A \cap B) \cap C = \{23^2, 29^2\} = \{\mathbf{529}, \mathbf{841}\}$$

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9.

$$
\begin{aligned}
f(3) \otimes g(5) &= (3+1) \otimes (2 \cdot 5 - 3) \\
&= 4 \otimes 7 \\
&= 2 \cdot 4 + 7 - 4 \cdot 7 \\
&= 8 + 7 - 28 \\
&= \mathbf{-13}
\end{aligned}
$$